Columbia University
in the City of New York

DEPARTMENT OF CIVIL ENGINEERING

FORCE AT A POINT IN THE INTERIOR
OF A SEMI-INFINITE SOLID

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Introduction

In a paper[1] under the same title, the solution of the linear equations of equilibrium of an elastic body was given for the case of a force acting at a point within an isotropic body bounded by a plane. The result was obtained by starting with Kelvin's solution for a force in an infinite body and guessing the nuclei of strain to add outside of the semi-infinite body so as to annul the tractions on the plane boundary. In the present paper it is shown how these results may be obtained, directly, by means of an application of potential theory.

Papkovitch Functions

In an isotropic elastic body in equilibrium, the displacement $\underset{\sim}{u}$ is governed by the equation[2]

$$\mu \nabla^2 \underset{\sim}{u} + \frac{\mu}{1-2\nu} \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\underset{\sim}{u} + \underset{\sim}{F} = 0 \, , \tag{1}$$

where μ is the shear modulus, ν is Poisson's ratio and $\underset{\sim}{F}$ is the body force per unit of volume.

For an isotropic body, the stress, $\underset{\sim}{\sigma}$, is related to the displacement by

$$\underset{\sim}{\sigma} = \lambda \, \underset{\sim}{\nabla}\cdot\underset{\sim}{u} \, \underset{\sim}{I} + \mu\left(\underset{\sim}{\nabla}\underset{\sim}{u} + \underset{\sim}{u}\underset{\sim}{\nabla}\right) \tag{2}$$

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[1] R. D. Mindlin, Physics, Vol. 7 (1936), pp. 195-202.

[2] For the vector notation used in this paper, see C. E. Weatherburn, Advanced Vector Analysis, G. Bell and Sons, Ltd., London, 1928.

where $\lambda = 2\mu\nu/(1-2\nu)$.

By Helmholtz's theorem[3], $\underset{\sim}{u}$ may be resolved into lamellar and sole-noidal components:

$$\underset{\sim}{u} = \nabla\varphi + \nabla\times\underset{\sim}{H}, \quad \nabla\cdot\underset{\sim}{H} = 0, \tag{3}$$

so that (1) may be written

$$\mu\nabla^2(\alpha\nabla\varphi + \nabla\times\underset{\sim}{H}) + \underset{\sim}{F} = 0 \tag{4}$$

where $\alpha = 2(1-\nu)/(1-2\nu)$.

The quantity in parenthesis in (4) is a vector, say, $\underset{\sim}{B} = \underset{\sim}{i}B_x + \underset{\sim}{j}B_y + \underset{\sim}{k}B_z$ i.e.,

$$\alpha\nabla\varphi + \nabla\times\underset{\sim}{H} = \underset{\sim}{B}, \tag{5}$$

$$\mu\nabla^2\underset{\sim}{B} = -\underset{\sim}{F}. \tag{6}$$

Operating on (5) with $\underset{\sim}{\nabla}\cdot$, we find

$$\alpha\nabla^2\varphi = \underset{\sim}{\nabla}\cdot\underset{\sim}{B} \tag{7}$$

the complete solution of which is

$$2\alpha\varphi = \underset{\sim}{r}\cdot\underset{\sim}{B} + \beta, \tag{8}$$

where β is a scalar function, which satisfies

$$\mu\nabla^2\beta = \underset{\sim}{r}\cdot\underset{\sim}{F}, \tag{9}$$

and $\underset{\sim}{r} = \underset{\sim}{i}x + \underset{\sim}{j}y + \underset{\sim}{k}z$ is the position vector.

Substituting (8) in (3) and eliminating $\underset{\sim}{\nabla}\times\underset{\sim}{H}$ by means of (5), there results

$$\underset{\sim}{u} = \underset{\sim}{B} - \frac{1}{4(1-\nu)}\underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{B} + \beta) \tag{10}$$

$$\mu\nabla^2\underset{\sim}{B} = -\underset{\sim}{F} \tag{11}$$

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[3] Weatherburn, p. 44.

$$\mu \nabla^2 \beta = \underset{\sim}{r} \cdot \underset{\sim}{F} \quad . \tag{12}$$

Thus the displacement is expressed in terms of the Papkovitch functions, $\underset{\sim}{B}$ and β , whose Laplacians are known if the body force $\underset{\sim}{F}$ is known.

The proof of completeness of the Papkovitch functions, given above, is an extension, to include the body force, of one given in a previous paper[4], where $\underset{\sim}{B}$ and β were called Papkovitch[5] functions after the originator of the solution (10) of the elasticity equations. Recent writers associate these functions with the name of Boussinesq[6], who introduced B_z and β , but employed functions of a different type where B_x and B_y could have been used.

Green's Formula

The value of a function V , at any point in a region, may be expressed in terms of its values at the boundary, its Laplacian and Green's function, G , for the region, by means of Green's formula[7]

$$-4\pi V = \int V \underset{\sim}{n} \cdot \underset{\sim}{\nabla} G \, dS + \int G \, \nabla^2 V \, dv \quad . \tag{13}$$

For the region $z \geqslant 0$, Green's function is

$$G = r_1^{-1} - r_2^{-1} \tag{14}$$

where

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[4] R. D. Mindlin, Bull. Am. Math. Soc., Vol. 42 (1936), pp. 373-376.

[5] P. F. Papkovitch, Comptes Rendus, Acad. des Sciences, Paris, Vol. 195 (1922), pp. 513-515 and 754-756.

[6] J. Boussinesq, "Application des Potentiels à l'Étude de l'Équilibre et du Mouvement des Solides Élastiques," Gauthier-Villars, Paris, 1885, pp. 63 and 72.

[7] Weatherburn, p. 34.

$$r_1^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \qquad (15)$$

$$r_2^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2 \qquad (16)$$

in which (x, y, z) are the coordinates of a point $P(x, y, z)$ in the region and (ξ, η, ζ), $(\xi, \eta, -\zeta)$ are the coordinates of a source point $Q(\xi, \eta, \zeta)$ and its image $Q'(\xi, \eta, -\zeta)$, respectively.

Force at a Point

Kelvin's definition of a force at a point takes the following form, in the present case. Consider a distribution of body forces $\underset{\sim}{F}$ in a closed region T within $z > 0$, with $\underset{\sim}{F} = 0$ outside T but within $z > 0$. Diminish T indefinitely, always enclosing the point $C(0, 0, c)$, but let

$$\lim_{T \to 0} \int_T \underset{\sim}{F}\, dv = \underset{\sim}{P} \qquad (17)$$

where $\underset{\sim}{P}$ is a constant force at C.

For later use, we note that the limit, as T approaches zero, of

$$Q(\xi, \eta, \zeta) = C(0, 0, c) \quad , \qquad (18)$$

$$Q'(\xi, \eta, -\zeta) = C'(0, 0, -c) \quad , \qquad (19)$$

$$r_1^2 = x^2 + y^2 + (z-c)^2 = R_1^2 \quad , \qquad (20)$$

$$r_2^2 = x^2 + y^2 + (z+c)^2 = R_2^2 \quad , \qquad (21)$$

$$G = \frac{1}{R_1} - \frac{1}{R_2} \quad , \qquad (22)$$

$$\frac{\partial G}{\partial \zeta} = -\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad , \qquad (23)$$

$$\frac{\partial^2 G}{\partial \zeta^2} = \frac{\partial^2}{\partial z^2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad , \qquad (24)$$

$$\frac{\partial G}{\partial \xi} = - \frac{\partial}{\partial x}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad . \tag{25}$$

Force Normal to the Plane Boundary

In this case we take $F_x = F_y = 0$ and $B_x = B_j = 0$. The remaining Papkovitch (in this case Boussinesq) functions, B_z and β, must satisfy the condition of vanishing traction on $z = 0$. Thus, we have, from (2) and (10), on $z = 0$

$$\sigma_{zz} = \frac{\mu}{2(1-\nu)}\left[2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2 \beta}{\partial z^2}\right] = 0 \quad , \tag{26}$$

$$\sigma_{zx} = \frac{\mu}{2(1-\nu)}\left[(1-2\nu)\frac{\partial B_z}{\partial x} - \frac{\partial^2 \beta}{\partial x \partial z}\right] = 0 \quad , \tag{27}$$

$$\sigma_{zy} = \frac{\mu}{2(1-\nu)}\left[(1-2\nu)\frac{\partial B_z}{\partial y} - \frac{\partial^2 \beta}{\partial y \partial z}\right] = 0. \tag{28}$$

The function in brackets in (26) is one whose Laplacian

$$\nabla^2\left[2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2 \beta}{\partial z^2}\right] = -\frac{2(1-\nu)}{\mu}\frac{\partial F_z}{\partial z} - \frac{1}{\mu}\frac{\partial^2(z F_z)}{\partial z^2} \tag{29}$$

is known throughout $z \geqslant 0$ and whose boundary value is zero. Hence, by (13),

$$2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2 \beta}{\partial z^2} = \frac{1}{4\pi\mu}\int G\left[2(1-\nu)\frac{\partial F_z}{\partial \zeta} + \frac{\partial^2(\zeta F_z)}{\partial \zeta^2}\right]dv \quad . \tag{30}$$

Now, integrating the first term in the volume integral by parts,

$$\iiint G \frac{\partial F_z}{\partial \zeta}\, d\xi\, d\eta\, d\zeta = \iint G\, F_z\, d\xi\, d\eta - \iiint F_z \frac{\partial G}{\partial \zeta}\, d\xi\, d\eta\, d\zeta. \tag{31}$$

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The surface integral in (31) vanishes because $F_z = G = 0$ on the boundary of the body. Then, by (17) and (23),

$$\lim_{T \to 0} \int G \frac{\partial F_z}{\partial \zeta} \, dv = P_z \frac{\partial}{\partial z} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad . \tag{32}$$

Similarly, integrating by parts twice and using (17), (18) and (24),

$$\lim_{T \to 0} \int G \frac{\partial^2 F_z}{\partial \zeta^2} \, dv = c P_z \frac{\partial^2}{\partial z^2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \quad . \tag{33}$$

Hence

$$2(1-\nu) B_z - \frac{\partial \beta}{\partial z} = \frac{P_z}{4\pi\mu} \left[2(1-\nu) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + c \frac{\partial}{\partial z} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right], \tag{34}$$

where one integration with respect to z has been performed. (The arbitrary function of x and y, thereby introduced, must vanish since $2(1-\nu) B_z - \partial\beta/\partial z$ must vanish as $z \to \infty$).

Returning to the boundary conditions, we note that (27) and (28) can be integrated with respect to x and y, respectively, so that, on $z = 0$,

$$(1-2\nu) B_z - \frac{\partial \beta}{\partial z} = 0 \quad . \tag{35}$$

The Laplacian

$$\nabla^2 \left[(1-2\nu) B_z - \frac{\partial \beta}{\partial z} \right] = -\frac{1-2\nu}{\mu} F_z - \frac{1}{\mu} \frac{\partial (z F_z)}{\partial z} \tag{36}$$

is known throughout $z \geqslant 0$ and hence, by (13),

$$(1-2\nu) B_z - \frac{\partial \beta}{\partial z} = \frac{1}{4\pi\mu} \int G \left[(1-2\nu) F_z + \frac{\partial}{\partial \zeta} (\zeta F_z) \right] dv \quad . \tag{37}$$

By the same process as before, using (17), (18), (22) and (23), we find

$$(1-2\nu)B_z - \frac{\partial \beta}{\partial z} = \frac{P_z}{4\pi\mu}\left[(1-2\nu)\left(\frac{1}{R} - \frac{1}{R_1}\right) + c\frac{\partial}{\partial z}\left(\frac{1}{R_1} + \frac{1}{R_1}\right)\right] . \quad (38)$$

Finally, from (34) and (38),

$$B_z = \frac{P_z}{4\pi\mu}\left[\frac{1}{R_1} + \frac{3-4\nu}{R_2} + \frac{2c(z+c)}{R_2^3}\right] \quad (39)$$

$$\beta = \frac{P_z}{4\pi\mu}\left[4(1-\nu)(1-2\nu)\log(R_2+z+c) - \frac{c}{R_1} - \frac{(3-4\nu)c}{R_2}\right] . \quad (40)$$

These two functions constitute the solution for the case of the force at $(0,0,c)$ normal to the plane boundary.

Force Parallel to the Plane Boundary

In this case we take $F_y = F_z = 0$ and $B_y = 0$. The boundary conditions then become, on $z = 0$,

$$\sigma_{zz} = \frac{\mu}{2(1-\nu)}\left[2(1-\nu)\frac{\partial B_z}{\partial z} + 2\nu\frac{\partial B_x}{\partial x} - x\frac{\partial^2 B_x}{\partial z^2} - \frac{\partial^2 \beta}{\partial z^2}\right] = 0 \quad (41)$$

$$\sigma_{zx} = \frac{\mu}{2(1-\nu)}\left[(1-2\nu)\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) - x\frac{\partial^2 B_x}{\partial x \partial z} - \frac{\partial^2 \beta}{\partial x \partial z}\right] = 0 \quad (42)$$

$$\sigma_{zy} = \frac{\mu}{2(1-\nu)}\left[(1-2\nu)\frac{\partial B_z}{\partial y} - x\frac{\partial^2 B_x}{\partial y \partial z} - \frac{\partial^2 \beta}{\partial y \partial z}\right] = 0 . \quad (43)$$

Differentiating (42) with respect to y and (43) with respect to x and subtracting, we find, on $z = 0$,

$$\frac{\partial^2 B_x}{\partial y \partial z} = 0 .$$

Hence, on $z = 0$,

$$\frac{\partial B_x}{\partial z} = 0 . \quad (44)$$

Also, in $z \geqslant 0$,

$$\nabla^2 \frac{\partial B_x}{\partial z} = -\frac{1}{\mu} \frac{\partial F_x}{\partial z} \qquad .$$

Hence, from (13),

$$\frac{\partial B_x}{\partial z} = \frac{1}{4\pi\mu} \int G \frac{\partial F_x}{\partial \zeta} dv$$

$$= \frac{P_x}{4\pi\mu} \frac{\partial}{\partial z} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

by (32). Thus

$$B_x = \frac{P_x}{4\pi\mu} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \qquad . \tag{45}$$

From (43), on $z = 0$,

$$(1-2\nu) B_z - \frac{\partial \beta}{\partial z} = x \frac{\partial B_x}{\partial z} = 0$$

by (44). Also, in $z \geqslant 0$

$$\nabla^2 \left[(1-2\nu) B_z - \frac{\partial \beta}{\partial z} \right] = - \frac{x}{\mu} \frac{\partial F_x}{\partial z} \qquad .$$

Hence, by (13)

$$(1-2\nu) B_z - \frac{\partial \beta}{\partial z} = \frac{1}{4\pi\mu} \int G \, \xi \, \frac{\partial F_x}{\partial \zeta} dv \qquad . \tag{46}$$

But the right hand side of (46) vanishes since $\xi \rightarrow 0$ as $T \rightarrow 0$. Hence

$$(1-2\nu) B_z - \frac{\partial \beta}{\partial z} = 0 \tag{47}$$

throughout the region $z \geqslant 0$.

From (41), on $z = 0$,

$$2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2 \beta}{\partial z^2} + 2\nu \frac{\partial B_x}{\partial x} - x\frac{\partial^2 B_x}{\partial z^2} = 0 \qquad (48)$$

Now, on $z = 0$, we have, from (45),

$$2\nu\frac{\partial B_x}{\partial x} - x\frac{\partial^2 B_x}{\partial z^2} = \frac{(1-2\nu)P_x x}{2\pi\mu R_o} - \frac{3P_x c^2 x}{2\pi\mu R_o^5}$$

where $R_o^2 = x^2 + y^2 + c^2$. But, on $z = 0$,

$$-(1-2\nu)\frac{\partial B_x}{\partial x} = \frac{(1-2\nu)P_x x}{2\pi\mu R_o}$$

and

$$\frac{P_x c}{2\pi\mu}\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{R_2}\right) = \frac{3P_x c^2 x}{2\pi\mu R_o^5} \qquad .$$

Hence, (48) may be rewritten as

$$\chi = 2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2 \beta}{\partial z^2} - (1-2\nu)\frac{\partial B_x}{\partial x} - \frac{P_x c}{2\pi\mu}\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{R_2}\right) = 0 \qquad (49)$$

on $z = 0$. The Laplacian of the left side of (49) is, in the region $z \gtrless 0$,

$$\nabla^2 \chi = -\frac{x}{\mu}\frac{\partial^2 F_x}{\partial z^2} + \frac{1-2\nu}{\mu}\frac{\partial F_x}{\partial x} \qquad .$$

Hence, from (13),

$$\chi = \frac{1}{4\pi\mu}\int G\left[\xi\frac{\partial^2 F_x}{\partial \xi^2} - (1-2\nu)\frac{\partial F_x}{\partial \xi}\right]dv \qquad . \qquad (50)$$

The first term in the integrand vanishes since $\xi \to 0$ as $T \to 0$. The second term is integrated by parts and the surface integral vanishes, leaving

$$\chi = \frac{1-2\nu}{4\pi\mu}\int F_x \frac{\partial G}{\partial \xi}\, dv$$

which, by (17) and (25), is

$$\chi = -\frac{(1-2\nu)R}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad . \tag{51}$$

Then, from (51), (49) and (45),

$$2(1-\nu)\frac{\partial B_z}{\partial z} - \frac{\partial^2\beta}{\partial z^2} = -\frac{(1-2\nu)P_x x}{2\pi\mu R_2^3} + \frac{3P_x c x(z+c)}{2\pi\mu R_2^5} \tag{52}$$

or

$$2(1-\nu)B_z - \frac{\partial\beta}{\partial z} = \frac{(1-2\nu)P_x x}{2\pi\mu R_2(R_2+z+c)} - \frac{P_x c x}{2\pi\mu R_2^3} \qquad . \tag{53}$$

Finally, from (53) and (47), we have

$$B_z = \frac{(1-2\nu)P_x x}{2\pi\mu R_2(R_2+z+c)} - \frac{P_x c x}{2\pi\mu R_2^3} \tag{54}$$

$$\beta = -\frac{(1-2\nu)^2 P_x x}{2\pi\mu(R_2+z+c)} + \frac{(1-2\nu)P_x c x}{2\pi\mu R_2(R_2+z+c)} \qquad . \tag{55}$$

These two functions, in addition to (45), comprise the solution for the case of the force at $(0,0,c)$ parallel to the plane boundary.

Comparison with Previous Results

The previous solution, mentioned in the Introduction, was given in terms of the Galerkin vector $\underset{\sim}{F}$ (not to be confused with the body force $\underset{\sim}{F}$ in the present paper).

For the case of a force normal to the plane boundary the solution obtained was

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$$F = \frac{P_z\, k}{8\pi(1-\nu)} \left\{ R_1 + \left[8\nu(1-\nu) - 1 \right] R_2 - 2c\,z/R_2 \right.$$
$$\left. + 4(1-2\nu)\left[(1-\nu)z - \nu c \right] \log(R_2 + z + c) \right\} \tag{56}$$

and, for the case of a force parallel to the plane boundary,

$$F = \frac{P_x\, i}{8\pi(1-\nu)} \left\{ R_1 + R_2 - 2c^2/R_2 \right.$$
$$\left. + 4(1-\nu)(1-2\nu)\left[(z+c)\log(R_2+z+c) - R_2 \right] \right\}$$
$$+ \frac{P_x\, k}{8\pi(1-\nu)} \left\{ 2cx/R_2 + 2(1-2\nu)x \log(R_2+z+c) \right\} . \tag{57}$$

The relation between the Galerkin and Papkovitch functions has been shown to be[4]

$$\mu B = (1-\nu)\nabla^2 F \tag{58}$$

$$\mu \beta = (1-\nu)\left(2\nabla \cdot F - r \cdot \nabla^2 F \right) \tag{59}$$

By inserting (56) and (57) in (58) and (59), it may be verified that the previous and present solutions are identical.